

Zero modes in the light-front coupled-cluster method

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(Dated: August 31, 2012)

Abstract

The light-front coupled-cluster method is a technique for solving Hamiltonian eigenvalue problems in light-front-quantized field theories. Its primary purpose is to provide a systematic sequence of solvable approximations to the original eigenvalue problem without the truncation of Fock space. Here we discuss the incorporation of zero modes, modes of zero longitudinal momentum, into the formalism of the method. The use and importance of these modes are illustrated in two-dimensional model field theories.

PACS numbers: 12.38.Lg, 11.15.Tk, 11.10.Ef

I. INTRODUCTION

One approach to the nonperturbative solution of a strongly interacting quantum field theory is that of Hamiltonian methods in light-front quantization [1, 2]. The Hamiltonian eigenvalue problem is formulated in Fock space as the fundamental equation

$$\mathcal{P}^- |\psi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+} |\psi(\underline{P})\rangle, \quad (1.1)$$

where \mathcal{P}^- is the light-front energy operator, conjugate to the light-front time $x^+ \equiv t + z$, and $\underline{P} = (P^+ \equiv E + P^z, \vec{P}_\perp = (P^x, P^y))$ is the light-front momentum, with P^+ conjugate to $x^- \equiv t - z$. The eigenstate $|\psi\rangle$ has mass M and is expanded in a Fock basis of eigenstates of \underline{P} and of particle number. The coefficients of the Fock states are the wave functions that describe the eigenstate. The eigenvalue problem (1.1) is equivalent to an infinite coupled system of integral equations for these wave functions.

Light-front quantization is quite convenient for this construction. In particular, the Fock-state expansion is well-defined, and there is a separation of external and internal momenta for the constituents, which leads to well-defined and boost-invariant wave functions [2]. It does, however, have the disadvantage that, except for modes of zero longitudinal momentum p^+ [3–5], there are no contributions to the vacuum, making the perturbative vacuum $|0\rangle$ the physical vacuum. The calculation of vacuum effects, such as symmetry breaking [4–9], is then a challenging aspect of light-front quantization, and the incorporation of zero modes can be important in some calculations. Here we consider zero modes in the context of the light-front coupled-cluster (LFCC) method [10, 11].

To have a finite calculation for an eigenstate, the usual step is a truncation of Fock space, to have a finite number of wave functions and a finite set of equations. This, however, leads to many difficulties, particularly uncanceled divergences [12]. The LFCC method is designed to avoid these difficulties by not truncating Fock space but instead restricting the relationships between wave functions in such a way as to produce a finite set of (nonlinear) equations. The mathematical structure of the method is closely related to that of the many-body coupled-cluster method [13] used in nuclear physics and physical chemistry [14].

The LFCC method constructs the eigenstate $|\psi\rangle$ from a valence state $|\phi\rangle$ with the smallest number of constituents and the exponentiation of an operator T that increases the particle numbers, to generate higher Fock states. The general form is

$$|\psi(\underline{P})\rangle = \sqrt{Z} e^T |\phi(\underline{P})\rangle, \quad (1.2)$$

with \sqrt{Z} a normalization factor. The eigenvalue problem is then converted to a valence eigenvalue problem

$$P_v \overline{\mathcal{P}}^- |\phi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+} |\phi(\underline{P})\rangle, \quad (1.3)$$

with $\overline{\mathcal{P}}^- \equiv e^{-T} \mathcal{P}^- e^T$ an effective Hamiltonian and P_v a projection onto the valence sector, and to an auxiliary equation for T

$$(1 - P_v) \overline{\mathcal{P}}^- |\phi(\underline{P})\rangle = 0. \quad (1.4)$$

The auxiliary equation is actually an infinite set of equations for the infinite set of terms in T , and as such we still have an exact representation of the original eigenvalue problem. The

approximations that lead to a finite set of equations, without truncating Fock space, are the truncation of T to a finite number of terms and a matching truncation of the projection $1 - P_\nu$ to generate the finite number of equations needed to solve for the terms in T .

Here we wish to take a step toward the inclusion of vacuum effects, such as spontaneous symmetry breaking in wrong-sign ϕ^4 , by introducing zero-mode contributions to the T operator in the LFCC method. We do this as a limiting procedure, with modes of momentum ϵ introduced at the start of a calculation and the limit $\epsilon \rightarrow 0$ taken at the end. The technique is developed in a series of examples; we discuss ϕ^3 theory [15], ϕ^4 theory [16–18], and the Wick–Cutkosky model [19] in Secs. II, III, and IV, respectively. A brief summary is given in Sec. V. Some details for singular terms are included in an Appendix.

II. ϕ^3 THEORY

The Lagrangian of ϕ^3 theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2 \phi^2 - \frac{\lambda}{3!}\phi^3. \quad (2.1)$$

From this, the two-dimensional light-front Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\partial_- \phi \partial_+ \phi - \mathcal{L} = \frac{1}{2}\mu^2 \phi^2 + \frac{\lambda}{3!}\phi^3. \quad (2.2)$$

The mode expansion for the field at zero light-front time is

$$\phi = \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ a(p^+) e^{-ip^+ x^-/2} + a^\dagger(p^+) e^{ip^+ x^-/2} \right\}, \quad (2.3)$$

with the modes quantized such that

$$[a(p^+), a^\dagger(p'^+)] = \delta(p^+ - p'^+). \quad (2.4)$$

The normal-ordered light-front Hamiltonian $\mathcal{P}^- = \mathcal{P}_{\text{free}}^- + \mathcal{P}_{\text{int}}^-$ is then specified by

$$\mathcal{P}_{\text{free}}^- = \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+) a(p^+) \quad (2.5)$$

and

$$\begin{aligned} \mathcal{P}_{\text{int}}^- = \frac{\lambda}{2} \int \frac{dx}{\sqrt{x(1-x)}} \int \frac{dp^+}{\sqrt{4\pi p^+}} & \left[a^\dagger(p^+) a(xp^+) a((1-x)p^+) \right. \\ & \left. + a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+) \right]. \end{aligned} \quad (2.6)$$

The form of $\mathcal{P}_{\text{int}}^-$, in terms of the momentum p^+ flowing through the operator and the longitudinal momentum fraction x , is particularly convenient for what is to follow. For a graphical representation of \mathcal{P}^- , see Fig. 1. The effective Hamiltonian $\overline{\mathcal{P}}^-$ of the LFCC method is computed from the Baker–Hausdorff expansion

$$\overline{\mathcal{P}}^- = e^{-T} \mathcal{P}^- e^T = \mathcal{P}^- + [\mathcal{P}^-, T] + \frac{1}{2!} [[\mathcal{P}^-, T], T] + \cdots, \quad (2.7)$$

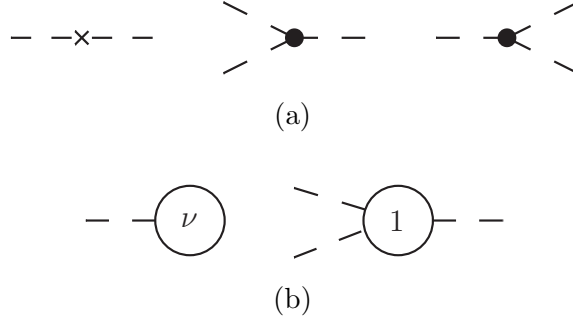


FIG. 1. Diagrammatic representation of (a) the light-front Hamiltonian \mathcal{P}^- and (b) the approximate T operator for ϕ^3 theory. A circle with a ν represents a zero-mode contribution; a circle with a 1 represents the function t_1 found in the T operator. The cross represents the kinetic energy contribution. External lines on the right represent annihilation operators; those on the left, creation operators; and internal lines, contractions.

given an approximation for T .

For ϕ^3 theory, the simplest approximation for T is a single boson emission process represented by

$$T_1 = \int dx \int dp^+ \sqrt{\frac{p^+}{4\pi}} t_1(x) a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+). \quad (2.8)$$

Here $t_1(x)$ is a function to be determined by the auxiliary equation (1.4). From the structure of T we can see that only the symmetric part of t_1 contributes; therefore, we can assume $t_1(x) = t_1(1-x)$. The corresponding truncation of the projection $1 - P_v$ is to include only states with one particle in addition to those in the valence state.

To include the zero mode, we introduce another operator to $T = T_0 + T_1$ as

$$T_0 = A_0 \epsilon^\alpha a^\dagger(\epsilon). \quad (2.9)$$

This creates a particle of momentum ϵ , which does violate momentum conservation for T ; however, the limit of $\epsilon \rightarrow 0$ will be taken at the end, so that T_0 is associated only with zero modes. The power α will be determined from the auxiliary equation, but the outcome can be anticipated by considering the corresponding transformation of the field:

$$e^{-T_0} \phi e^{T_0} = \phi + [\phi, T_0] = \phi + \frac{A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}}, \quad (2.10)$$

which indicates $\alpha = 1/2$ and a shift of $\nu \equiv A_0/\sqrt{4\pi}$ for the field. A graphical representation of T is given in Fig. 1.

A calculation of the terms in the Baker–Hausdorff expansion (2.7) then determines the effective Hamiltonian. Only a finite number of terms will contribute to the eigenvalue problem, because we need only terms that change the particle number by no more than one; any more than this would go beyond the truncation of the projection $(1 - P_v)$. We will consider only the vacuum and the one-particle state as valence states, and, therefore, terms in $\overline{\mathcal{P}}^-$ with more than one annihilation operator will also be neglected; however, terms with

more than one do need to be kept in intermediate calculations of commutators, because higher-order commutators can reduce the total number of annihilation operators.

We compute the following commutators for T_0 :

$$[\mathcal{P}_{\text{free}}^-, T_0] = \frac{\mu^2}{\epsilon} A_0 \epsilon^\alpha a^\dagger(\epsilon), \quad (2.11)$$

$$[[\mathcal{P}_{\text{free}}^-, T_0], T_0] = 0, \quad (2.12)$$

$$[\mathcal{P}_{\text{int}}^-, T_0] = \lambda \int \frac{dp^+}{\sqrt{4\pi p^+}} \frac{A_0 \epsilon^\alpha}{\sqrt{\epsilon(p^+ - \epsilon)}} a^\dagger(p^+) a(p^+ - \epsilon), \quad (2.13)$$

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_0] = \lambda \frac{A_0^2 \epsilon^{2\alpha}}{\epsilon \sqrt{4\pi \epsilon}} a^\dagger(\epsilon). \quad (2.14)$$

For $[\mathcal{P}_{\text{int}}^-, T_0]$, we keep only the leading term as $\epsilon \rightarrow 0$; this leaves

$$[\mathcal{P}_{\text{int}}^-, T_0] \rightarrow \lambda \int \frac{dp^+}{p^+} \frac{A_0 \epsilon^\alpha}{\sqrt{4\pi \epsilon}} a^\dagger(p^+) a(p^+), \quad (2.15)$$

which we use in the computation of $[[\mathcal{P}_{\text{int}}^-, T_0], T_0]$. There is a subtle aspect of the $\epsilon \rightarrow 0$ limit, in that the integration range of p^+ includes a region where it is not large compared to ϵ ; a more careful calculation is described in the Appendix, with the same result.

The commutators for T_1 are

$$[\mathcal{P}_{\text{free}}^-, T_1] = \int \frac{dx}{\sqrt{x(1-x)}} \int \frac{dp^+}{\sqrt{4\pi p^+}} t_1(x) \left[\frac{\mu^2}{x(1-x)} - \mu^2 \right] \times a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+), \quad (2.16)$$

$$\begin{aligned} [\mathcal{P}_{\text{int}}^-, T_1] = & \int dp^+ \frac{I}{p^+} a^\dagger(p^+) a(p^+) \\ & - \frac{\lambda}{2} \int \frac{dx}{\sqrt{x(1-x)}} \int dx' t_1(x') \\ & \times \int \frac{dp^+}{4\pi} a^\dagger(x'p^+) a^\dagger((1-x')p^+) a(xp^+) a((1-x)p^+) \\ & + 2\lambda \int \frac{dx}{\sqrt{x(1-x)}} \int dx' t_1(x') \sqrt{\frac{1-x}{1-x'}} \int \frac{dp^+}{\sqrt{4\pi}} \\ & \times \int \frac{dp'^+}{\sqrt{4\pi}} \delta((1-x)p^+ - (1-x')p'^+) a^\dagger(p^+) a^\dagger(x'p'^+) a(xp^+) a(p'^+), \\ [[\mathcal{P}_{\text{int}}^-, T_1], T_1] = & \int dx \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ t_1(x) \left[\frac{I}{x(1-x)} - 2I \right] + V(x) \right\} \\ & \times a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+), \end{aligned} \quad (2.17)$$

$$[[\mathcal{P}_{\text{int}}^-, T_1], T_1] = \int dx \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ t_1(x) \left[\frac{I}{x(1-x)} - 2I \right] + V(x) \right\} \times a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+), \quad (2.18)$$

with the self-energy integral

$$I \equiv \frac{\lambda}{4\pi} \int \frac{dx t_1(x)}{\sqrt{x(1-x)}} \quad (2.19)$$

and vertex function

$$V(x) = \frac{1}{2} \left[\tilde{V}(x) + \tilde{V}(1-x) \right], \quad (2.20)$$

$$\tilde{V}(x) \equiv \frac{\lambda}{\pi} \int_x^1 dy \frac{t_1(x/y) t_1(y)}{\sqrt{y(1-y)(y-x)(1-x)}}. \quad (2.21)$$

Here and below we have taken advantage of the symmetry $t_1(x) = t_1(1-x)$ described following Eq. (2.8). The double commutators that mix T_0 and T_1 are

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_1] = \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \int dx \int \frac{dp^+}{\sqrt{4\pi p^+}} t_1(x) \left[\frac{1}{x(1-x)} - 1 \right] \times a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+) \quad (2.22)$$

$$[[\mathcal{P}_{\text{int}}^-, T_1], T_0] = I \frac{A_0 \epsilon^\alpha}{\epsilon} a^\dagger(\epsilon) + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \int dx \int \frac{dp^+}{\sqrt{4\pi p^+}} t_1(x) \left[\frac{1}{x(1-x)} - 1 \right] \times a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+). \quad (2.23)$$

That the self-energy integral is the same in all terms is one of the hallmarks of the LFCC method. For calculations with Fock-space truncation, the self-energy contributions are sector and spectator dependent [12].

From these commutators we build the expression for $\overline{\mathcal{P}}^-$ as

$$\begin{aligned} \overline{\mathcal{P}}^- = & A_0 \frac{\epsilon^\alpha}{\epsilon} \left[\mu^2 + \frac{1}{2} I + \frac{1}{2} \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right] a^\dagger(\epsilon) \\ & + \int \frac{dp^+}{p^+} \left[\mu^2 + I + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right] a^\dagger(p^+) a(p^+) \\ & + \int dx \int \frac{dp^+}{\sqrt{4\pi p^+}} a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+) \\ & \times \left\{ \left(\mu^2 + I + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right) \left[\frac{1}{x(1-x)} - 1 \right] t_1(x) \right. \\ & \quad \left. + \frac{\lambda}{2} \frac{1}{\sqrt{x(1-x)}} + \frac{1}{2} V(x) - \frac{1}{2} \frac{I}{x(1-x)} t_1(x) \right\}. \end{aligned} \quad (2.24)$$

We have kept only those terms that do not annihilate the one-particle Fock state. A graphical representation is given in Fig. 2

For a one-particle valence state $|\phi\rangle = a^\dagger(P^+) |0\rangle$, the eigenvalue problem $P_v \overline{\mathcal{P}}^- |\phi\rangle = \frac{M^2}{P^+} |\phi\rangle$ is

$$\left[\mu^2 + I + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right] |\phi\rangle = M^2 |\phi\rangle. \quad (2.25)$$

The auxiliary equation (1.4) in general includes two and more particle sectors. For our approximation, $1 - P_v$ is truncated to two-particle states only, and (1.4) becomes

$$\begin{aligned} & A_0 \frac{\epsilon^\alpha}{\epsilon} \left[\mu^2 + \frac{1}{2} I + \frac{1}{2} \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right] a^\dagger(\epsilon) a^\dagger(P^+) |0\rangle \\ & + \int \frac{dx}{\sqrt{4\pi P^+}} a^\dagger(xP^+) a^\dagger((1-x)P^+) |0\rangle \\ & \times \left\{ \left(\mu^2 + I + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right) \left[\frac{1}{x(1-x)} - 1 \right] t_1(x) \right. \\ & \quad \left. + \frac{\lambda}{2} \frac{1}{\sqrt{x(1-x)}} + \frac{1}{2} V(x) - \frac{1}{2} \frac{I}{x(1-x)} t_1(x) \right\} = 0. \end{aligned} \quad (2.26)$$

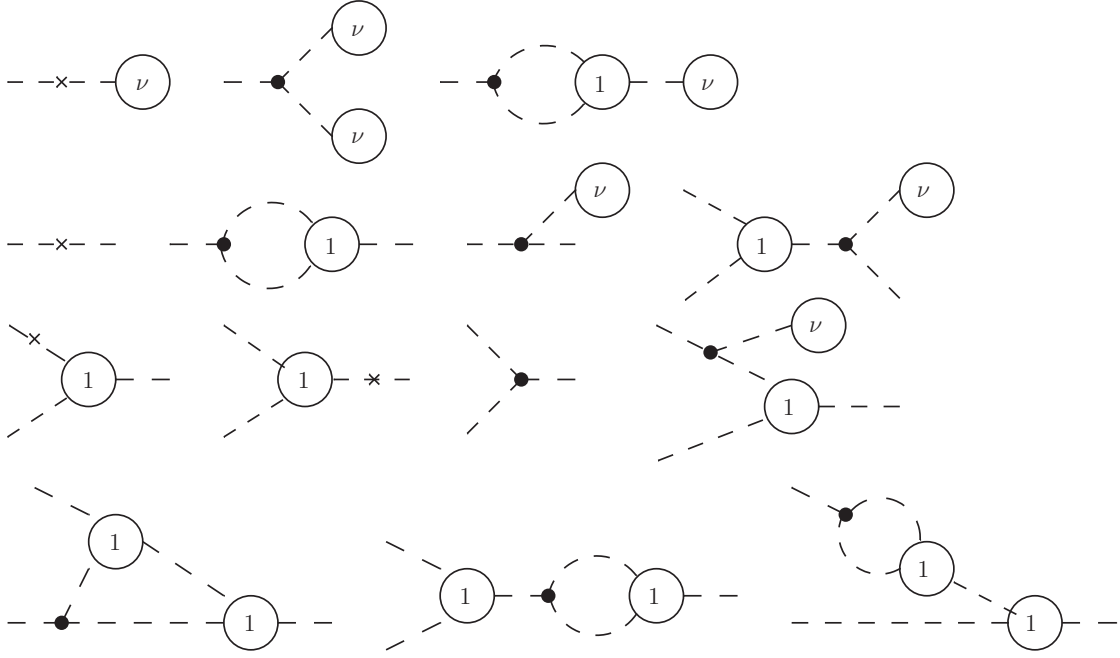


FIG. 2. Same as Fig. 1, but for the one-particle terms in the effective Hamiltonian $\overline{\mathcal{P}}^-$.

This equation contains two different contributions, the first with total momentum $P^+ + \epsilon$ and the second with total momentum P^+ . Projection onto $a^\dagger(\epsilon)a^\dagger(P^+)|0\rangle$ gives

$$A_0 \left[\mu^2 + \frac{1}{2}I + \frac{1}{2} \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right] = 0, \quad (2.27)$$

and onto $a^\dagger(xP^+)a^\dagger((1-x)P^+)|0\rangle$ yields

$$\left(\mu^2 + I + \frac{\lambda A_0 \epsilon^\alpha}{\sqrt{4\pi\epsilon}} \right) \left[\frac{1}{x(1-x)} - 1 \right] t_1(x) + \frac{\lambda}{2} \frac{1}{\sqrt{x(1-x)}} + \frac{1}{2} V(x) - \frac{1}{2} \frac{I}{x(1-x)} t_1(x) = 0. \quad (2.28)$$

From the first projection (2.27), we conclude that $A_0 = 0$ or $\alpha = 1/2$ and¹

$$\nu = \frac{A_0}{\sqrt{4\pi}} = -(2\mu^2 + I)/\lambda. \quad (2.29)$$

The valence eigenvalue problem is then trivially solved as $M^2 = -\mu^2$, and the auxiliary equation (2.28) reduces to

$$-\mu^2 \left[\frac{1}{x(1-x)} - 1 \right] t_1(x) + \frac{\lambda}{2} \frac{1}{\sqrt{x(1-x)}} + \frac{1}{2} V(x) - \frac{1}{2} \frac{I}{x(1-x)} t_1(x) = 0. \quad (2.30)$$

Because the physical one-particle eigenstate is computed to have a negative mass squared of $-\mu^2$ for any coupling value, one can construct multi-particle eigenstates of well-separated particles to have a Hamiltonian eigenvalue of any large multiple of $-\mu^2$. This is consistent with the demonstration by Baym [20] that the spectrum of ϕ^3 is unbounded from below. Without the inclusion of zero-mode contributions, this would not be apparent in the LFCC method.

¹ We obtain the same result from a vacuum valence state and the action of $(1 - P_v)\overline{\mathcal{P}}^-$ on the vacuum.

III. ϕ^4 THEORY

The Lagrangian and light-front Hamiltonian density for ϕ^4 theory are

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \quad (3.1)$$

and

$$\mathcal{H} = \frac{1}{2}\mu^2 \phi^2 + \frac{\lambda}{4!}\phi^4. \quad (3.2)$$

The mode expansion for the field ϕ is the same as (2.3) for ϕ^3 theory. We again split the light-front Hamiltonian \mathcal{P}^- into two parts

$$\mathcal{P}_{\text{free}}^- = \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+) a(p^+) \quad (3.3)$$

and

$$\begin{aligned} \mathcal{P}_{\text{int}}^- = & \frac{\lambda}{6} \int \frac{dx_1 dx_2 dx_3}{\sqrt{x_1 x_2 x_3}} \delta(1 - \sum x_i) \int \frac{dp^+}{4\pi} [a^\dagger(p^+) a(x_1 p^+) a(x_2 p^+) a(x_3 p^+) \\ & + a^\dagger(x_1 p^+) a^\dagger(x_2 p^+) a^\dagger(x_3 p^+) a(p^+)] \\ & + \frac{\lambda}{4} \int \frac{dx dx'}{\sqrt{x(1-x)x'(1-x')}} \int \frac{dp^+}{4\pi} a^\dagger(x p^+) a^\dagger((1-x)p^+) a(x' p^+) a((1-x')p^+). \end{aligned} \quad (3.4)$$

A graphical representation is shown in Fig. 3.

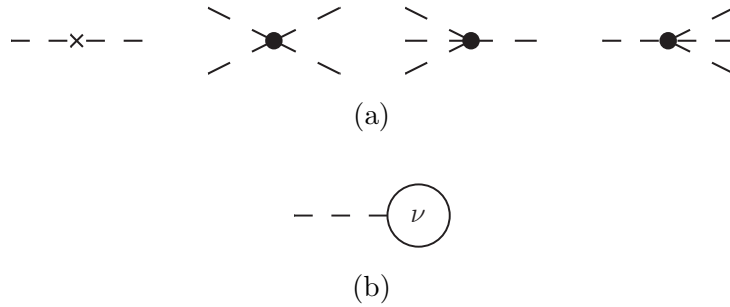


FIG. 3. Same as Fig. 1, but for ϕ^4 theory. Only the zero-mode term is included in (b).

We focus on the zero-mode contributions to a vacuum valence state and consider the T operator

$$T_0 = \nu \sqrt{4\pi\epsilon} a^\dagger(\epsilon). \quad (3.5)$$

As before in (2.10), this corresponds to a shift of the field ϕ by a constant ν . The relevant

commutators with \mathcal{P}^- are

$$[\mathcal{P}_{\text{free}}^-, T_0] = \nu \sqrt{\frac{4\pi}{\epsilon}} \mu^2 a^\dagger(\epsilon), \quad (3.6)$$

$$[\mathcal{P}_{\text{int}}^-, T_0] = \nu \frac{\lambda}{2} \int \frac{dx}{\sqrt{x(1-x)}} \int \frac{dp^+}{\sqrt{4\pi p^+}} [a^\dagger(p^+) a(xp^+) a((1-x)p^+) + a^\dagger(xp^+) a^\dagger((1-x)p^+) a(p^+)], \quad (3.7)$$

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_0] = \nu^2 \lambda \int \frac{dp^+}{p^+} a^\dagger(p^+) a(p^+), \quad (3.8)$$

$$[[[\mathcal{P}_{\text{int}}^-, T_0], T_0], T_0] = \nu^3 \lambda \sqrt{\frac{4\pi}{\epsilon}} a^\dagger(\epsilon), \quad (3.9)$$

where we keep only terms that will contribute to a projection $1 - P_v$ onto states with one particle more than in the valence state. From these commutators we construct the effective Hamiltonian for the zero modes

$$\overline{\mathcal{P}^-}|_{\text{ZM}} = \nu(\mu^2 + \nu^2 \lambda/6) \sqrt{\frac{4\pi}{\epsilon}} a^\dagger(\epsilon). \quad (3.10)$$

Figure 4 provides a graphical representation. The first graph corresponds to the commutator

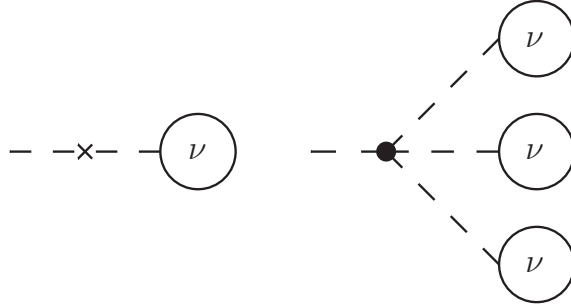


FIG. 4. Same as Fig. 2, but for ϕ^4 theory. Only zero-mode terms are included..

in (3.6) and the first term of $\overline{\mathcal{P}^-}|_{\text{ZM}}$; the second to the triple commutator in (3.9) and the second term of $\overline{\mathcal{P}^-}|_{\text{ZM}}$. Each commutator with \mathcal{P}^- contracts one zero-mode creation operator with one annihilation operator in \mathcal{P}^- .

For a vacuum valence state, the valence eigenvalue problem is trivial, because $P_v \overline{\mathcal{P}^-}|0\rangle = 0$. The auxiliary equation (1.4) is projected onto the one-particle sector and becomes

$$\nu(\mu^2 + \nu^2 \lambda/6) \sqrt{\frac{4\pi}{\epsilon}} a^\dagger(\epsilon)|0\rangle = 0. \quad (3.11)$$

This implies that either ν is zero or

$$\mu^2 + \frac{1}{6} \lambda \nu^2 = 0. \quad (3.12)$$

If we now consider the wrong-sign case, with $\mu^2 \rightarrow -\mu^2$, we find $\nu = \pm \sqrt{6\lambda}/\mu$, which corresponds to the shift of the field ϕ that brings the Hamiltonian density to a minimum.

Thus, the inclusion of a zero mode in the LFCC T operator allows for the necessary shift in the field. Also, as can be seen from (3.7), the effective Hamiltonian will have terms that mix Fock states with odd and even numbers of particles, which is characteristic of broken symmetry.

IV. WICK–CUTKOSKY MODEL

To illustrate what happens in a more complicated theory, we consider the Wick–Cutkosky model [19] of a charged scalar coupled to a neutral scalar. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2 \phi^2 + (\partial_\mu \chi)^2 - m^2 \chi^2 - g\phi|\chi|^2, \quad (4.1)$$

where ϕ is the neutral scalar field and χ the complex charged scalar field. The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\mu^2 \phi^2 + m^2 |\chi|^2 + g\phi|\chi|^2. \quad (4.2)$$

The mode expansion for the field ϕ is the same as (2.3) for ϕ^3 theory; the mode expansion for χ is

$$\chi = \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ c_+(p^+) e^{-ip^+ x^-/2} + c_-^\dagger(p^+) e^{ip^+ x^-/2} \right\}, \quad (4.3)$$

with c_\pm^\dagger the creation operator for the positive (negative) charge. The nonzero commutation relation is

$$[c_\pm(p^+), c_\pm^\dagger(p'^+)] = \delta(p^+ - p'^+). \quad (4.4)$$

The free and interacting parts of the light-front Hamiltonian \mathcal{P}^- are

$$\mathcal{P}_{\text{free}}^- = \int \frac{dp^+}{p^+} \left[\mu^2 a^\dagger(p^+) a(p^+) + m^2 c_+^\dagger(p^+) c_+(p^+) + m^2 c_-^\dagger(p^+) c_-(p^+) \right] \quad (4.5)$$

and

$$\begin{aligned} \mathcal{P}_{\text{int}}^- = g \int \frac{dx}{\sqrt{x(1-x)}} \int \frac{dp^+}{\sqrt{4\pi p^+}} & \left[a^\dagger(p^+) c_+(xp^+) c_-((1-x)p^+) \right. \\ & + c_+^\dagger(xp^+) c_-^\dagger((1-x)p^+) a(p^+) \\ & + a^\dagger(xp^+) \left(c_+^\dagger((1-x)p^+) c_+(p^+) + c_-^\dagger((1-x)p^+) c_-(p^+) \right) \\ & \left. + \left(c_+^\dagger(p^+) c_+((1-x)p^+) + c_-^\dagger(p^+) c_-((1-x)p^+) \right) a(xp^+) \right]. \end{aligned} \quad (4.6)$$

A graphical representation is given in Fig. 5.

We again focus on the zero-mode contributions and consider the T operator

$$T_0 = \nu \sqrt{4\pi\epsilon} a^\dagger(\epsilon) + 4\pi\epsilon^\beta \int dx t_0(x) c_+^\dagger(x\epsilon) c_-^\dagger((1-x)\epsilon). \quad (4.7)$$

The first term creates a neutral-scalar zero mode in the $\epsilon \rightarrow 0$ limit; the second creates a neutral pair of charged zero modes. The first term again corresponds to a shift of the field

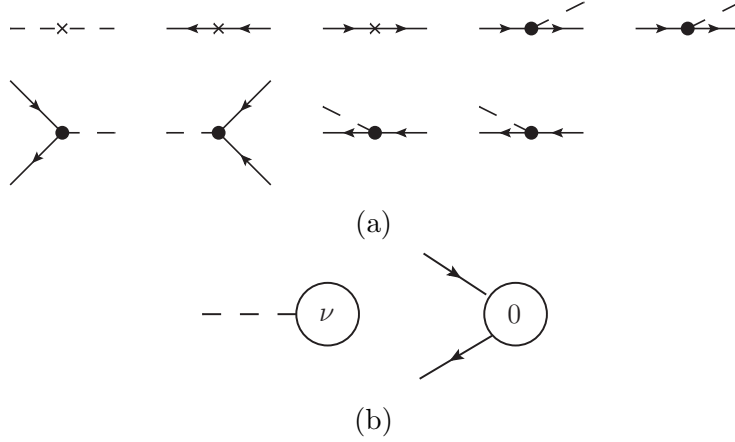


FIG. 5. Same as Fig. 3, but for the Wick–Cutkosky model. The charged scalars are represented by solid lines, with the arrow to the left (right) for positive (negative) charge; the neutral scalars are represented by dashed lines. The zero-mode contributions are labeled by ν for the neutral scalar and by 0 for the charged pair.

ϕ by a constant ν . The relevant commutators with \mathcal{P}^- are

$$[\mathcal{P}_{\text{free}}^-, T_0] = \nu \sqrt{\frac{4\pi}{\epsilon}} \mu^2 a^\dagger(\epsilon) + 4\pi m^2 \frac{\epsilon^\beta}{\epsilon} \int \frac{dx t_0(x)}{x(1-x)} c_+^\dagger(x\epsilon) c_-^\dagger((1-x)\epsilon), \quad (4.8)$$

$$\begin{aligned} [\mathcal{P}_{\text{int}}^-, T_0] = & \nu g \int \frac{dx}{\sqrt{x(1-x)}} c_+^\dagger(x\epsilon) c_-^\dagger((1-x)\epsilon) + g \frac{4\pi\epsilon^\beta}{\epsilon\sqrt{4\pi\epsilon}} \int \frac{dx t_0(x)}{\sqrt{x(1-x)}} a^\dagger(\epsilon) \\ & + \nu g \int \frac{dp^+}{p^+} (c_+^\dagger(p^+) c_+(p^+) + c_-^\dagger(p^+) c_-(p^+)), \end{aligned} \quad (4.9)$$

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_0] = 4\pi\nu g \frac{\epsilon^\beta}{\epsilon} \int \frac{dx t_0(x)}{x(1-x)} c_+^\dagger(x\epsilon) c_-^\dagger((1-x)\epsilon). \quad (4.10)$$

These yield an effective Hamiltonian of

$$\begin{aligned} \overline{\mathcal{P}}^-|_{\text{ZM}} = & \sqrt{\frac{4\pi}{\epsilon}} \left(\nu\mu^2 + g \frac{\epsilon^\beta}{\epsilon} \int \frac{dx t_0(x)}{\sqrt{x(1-x)}} \right) a^\dagger(\epsilon) \\ & + \int dx \left(4\pi \frac{\epsilon^\beta}{\epsilon} \frac{m^2 + \nu g}{x(1-x)} t_0(x) + \frac{\nu g}{\sqrt{x(1-x)}} \right) c_+^\dagger(x\epsilon) c_-^\dagger((1-x)\epsilon). \end{aligned} \quad (4.11)$$

Figure 6 shows a graphical representation.

To have enough equations to solve for the unknowns ν and t_0 in the T operator, we must project the auxiliary equation (1.4) onto one and two-particle sectors orthogonal to the vacuum valence state. This yields

$$\nu\mu^2 + g \frac{\epsilon^\beta}{\epsilon} \int \frac{dx t_0(x)}{\sqrt{x(1-x)}} = 0 \quad (4.12)$$

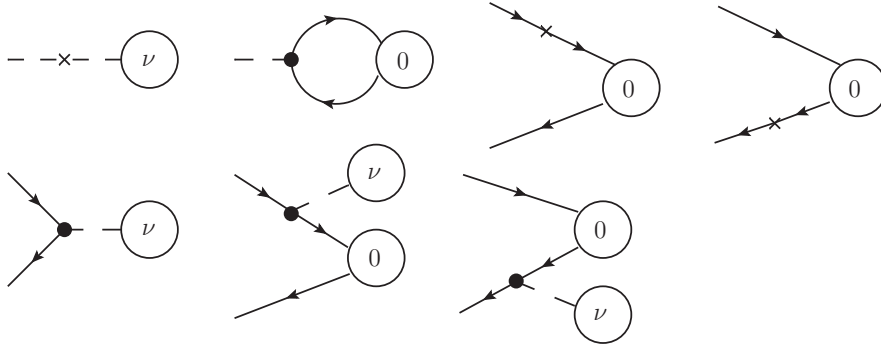


FIG. 6. Same as Fig. 5, but for the zero-mode terms in the effective Hamiltonian $\overline{\mathcal{P}^-}$.

and

$$4\pi \frac{\epsilon^\beta m^2 + \nu g}{\epsilon x(1-x)} t_0(x) + \frac{\nu g}{\sqrt{x(1-x)}} = 0. \quad (4.13)$$

A consistent solution is obtained if either $\nu = 0$ and $t_0 = 0$, which is a trivial solution without zero modes, or

$$\nu = \frac{g}{4\pi\mu^2} - \frac{m^2}{g}, \quad t_0(x) = -\frac{\mu^2\nu}{g} \sqrt{x(1-x)}, \quad (4.14)$$

with $\beta = 1$, which shifts the field. We can speculate that this solution will lead to the expected unbounded spectrum. A complete calculation of the eigenstates would require addition of non-zero-mode terms to the T operator and calculation of various commutators. Because of the relative complexity of the Wick–Cutkosky model, compared to ϕ^3 theory, this is nontrivial and beyond the scope of the present work. However, we have shown that nontrivial zero-mode contributions to the T operator and, therefore, to the eigenstates can be computed in the LFCC approach.

V. SUMMARY

We have considered various examples of two-dimensional scalar theories where zero modes can play a role in the calculation of light-front Hamiltonian eigenstates. In each case, the LFCC method is able to incorporate the zero modes in a sensible way. To do this, the T operator must include terms that allow creation of modes with infinitesimal longitudinal momentum ϵ , with the limit of $\epsilon \rightarrow 0$ taken at the end.

Although the examples are limited to two dimensions, there is no particular restriction on a direct extension to three or four dimensions. The zero-mode terms would include a dependence on transverse momenta. With these tools in place, one can use the LFCC to explore symmetry breaking nonperturbatively.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy through Contract No. DE-FG02-98ER41087.

Appendix A: Singular terms

For a more careful treatment of singular terms in the limit of $\epsilon \rightarrow 0$ for ϕ^3 theory, we begin with a more general zero-mode operator of the form

$$T_0 = \int dp^+ g(p^+) a^\dagger(p^+), \quad (\text{A1})$$

where g is limited to have most, if not all, of its support near zero. The commutators with \mathcal{P}^- are

$$[\mathcal{P}_{\text{free}}^-, T_0] = \int dp^+ \frac{\mu^2}{p^+} g(p^+) a^\dagger(p^+) \quad (\text{A2})$$

$$[\mathcal{P}_{\text{int}}^-, T_0] = \lambda \int \frac{dp^+}{\sqrt{4\pi p^+}} \int_0^{p^+} \frac{g(p^+) dp'^+}{\sqrt{p'^+(p^+ - p'^+)}} a^\dagger(p^+) a(p^+ - p'^+) \quad (\text{A3})$$

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_0] = \lambda \int \frac{dp^+}{\sqrt{4\pi p^+}} \int_0^{p^+} \frac{g(p^+) dp'^+}{\sqrt{p'^+(p^+ - p'^+)}} g(p^+ - p'^+) a^\dagger(p^+), \quad (\text{A4})$$

$$[[\mathcal{P}_{\text{int}}^-, T_0], T_0]|_{\text{ZM}} = \int dp^+ \frac{I}{p^+} g(p^+) a^\dagger(p^+), \quad (\text{A5})$$

where I is the self-energy integral (2.19). The first term in (2.24) becomes

$$\overline{\mathcal{P}^-}|_{\text{ZM}} = \int dp^+ \left[\frac{\mu^2 + I/2}{p^+} + \frac{\lambda/2}{\sqrt{4\pi p^+}} \int_0^{p^+} dp'^+ \frac{g(p'^+) g(p^+ - p'^+)}{\sqrt{p'^+(p^+ - p'^+)}} \right] a^\dagger(p^+). \quad (\text{A6})$$

The auxiliary equation (1.4), relative to a vacuum valence state, reduces to

$$\frac{\mu^2 + I/2}{p^+} + \frac{\lambda/2}{\sqrt{4\pi p^+}} \int_0^{p^+} dp'^+ \frac{g(p'^+) g(p^+ - p'^+)}{\sqrt{p'^+(p^+ - p'^+)}} = 0. \quad (\text{A7})$$

We replace g by $\tilde{g}(p^+) \equiv g(p^+)/\sqrt{p^+}$ to obtain

$$(\mu^2 + \frac{1}{2}I)\tilde{g}(p^+) = -\frac{\lambda/2}{\sqrt{4\pi}} \int_0^{p^+} dp'^+ \tilde{g}(p'^+) \tilde{g}(p^+ - p'^+). \quad (\text{A8})$$

To solve this equation, we introduce the Laplace transform $G(s) = \int_0^\infty e^{-sp^+} \tilde{g}(p^+) dp^+$ and make use of the fact that the Laplace transform of the convolution on the right-hand side is just the product of the transforms. This leaves

$$(\mu^2 + \frac{1}{2}I)G(s) = -\frac{\lambda/2}{\sqrt{4\pi}} G^2(s), \quad (\text{A9})$$

for which the solution is either $G = 0$ or $G = -\frac{\sqrt{4\pi}}{\lambda/2}(\mu^2 + \frac{1}{2}I)$, a constant. The inverse Laplace transform of a constant is, of course, a Dirac delta function; therefore, we obtain

$$\tilde{g}(p^+) = -\frac{\sqrt{4\pi}}{\lambda/2}(\mu^2 + \frac{1}{2}I)\delta(p^+), \quad (\text{A10})$$

which, given the $\epsilon \rightarrow 0$ limit to be taken, we can interpret as fixing g to be

$$g(p^+) = -\frac{\sqrt{4\pi\epsilon}}{\lambda/2}(\mu^2 + \frac{1}{2}I)\delta(p^+ - \epsilon). \quad (\text{A11})$$

This corresponds to the simpler expression (2.9) used for T_0 in Sec. II, with $\alpha = 1/2$ and A_0 given by the solution in (2.29).

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